

Gaussian Binomial Coefficients

We study the number of k -dimensional subspaces of \mathbb{F}_p^n , given by the Gaussian binomial coefficients. These are the finite-field analogue of binomial coefficients, counting subspaces instead of subsets.

Theorem 1. *The number of k -dimensional subspaces of \mathbb{F}_p^n is given by*

$$\binom{n}{k}_p = \frac{(p^n - 1)(p^n - p) \cdots (p^n - p^{k-1})}{(p^k - 1)(p^k - p) \cdots (p^k - p^{k-1})}.$$

Proof. We first count the number of ordered k -tuples of linearly independent vectors (let N) in \mathbb{F}_p^n . The first vector can be any nonzero vector, giving $p^n - 1$ choices. The second vector must lie outside the span of the first, giving $p^n - p$ choices. In general, the i -th vector must lie outside an $(i - 1)$ -dimensional subspace, giving $p^n - p^{i-1}$ choices. Thus,

$$N = (p^n - 1)(p^n - p) \cdots (p^n - p^{k-1})$$

Each such ordered k -tuple forms an ordered basis of a k -dimensional subspace of \mathbb{F}_p^n . However, a given subspace has many ordered bases. Let D denote the number of ordered bases of a fixed k -dimensional vector space over \mathbb{F}_p . Using same counting argument,

$$D = (p^k - 1)(p^k - p) \cdots (p^k - p^{k-1})$$

Since every k -dimensional subspace has exactly D ordered bases, each subspace is counted exactly D times in N . Therefore, the total number of k -dimensional subspaces is N/D which gives the desired formula. \square