Expander Graphs

This Daily Computer introduces expander graphs and their important properties. Expander graphs are *sparse* yet well *connected*.

Let $G = (V, E)$ be an undirected, d-regular graph (each vertex has degree d) such that $|V| = n$. For $S, T \subseteq V$ denote by $E(S, T)$ the set of edges connecting vertices in S and T.

Combinatorial Viewpoint. The *Expansion Ratio* of G, denoted by $h(G)$, is defined as,

$$h(G) = \min_{\substack{S \subseteq V \\ |S| \leq n/2}} \frac{|\delta(S)|}{|S|}$$

where $\delta(S)$ is the *Edge Boundary* of set S defined by $\delta(S) := E(S, V \setminus S)$ that is, the set of edges connecting S to its complement.

Definition 1 (Expander Graphs)**.** A sequence of d-regular (SPARSE) graphs $\{G_i\}_{i \in \mathbb{N}}$ with increasing size is called a *Family of Expander Graphs* if there exists $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ (CONNECTED) for all i.

Intuitively, expander graphs have strong connectivity since every small set of vertices has many edges leaving it.

Spectral Viewpoint. Let $A = A(G)$ denote the adjacency matrix of G. Since A is symmetric, all eigenvalues are real. Let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

be the eigenvalues of A, also called the *Spectrum* of G. For a d-regular graph, $\lambda_1 = d$. Define $\lambda := \max\{|\lambda_2|, |\lambda_n|\}$.

Lemma 1 (Expander Mixing Lemma)**.** *For all subsets $S, T \subseteq V$,*

$$\left| E(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

Note that, the quantity $\frac{d|S||T|}{n}$ is precisely the expected number of edges between S and T in a random d-regular graph. Therefore, if λ is small, then the number of edges between any two sets is close to what we would expect in a random graph. Thus, graphs with small λ exhibit *pseudorandomness*: although deterministic, their edge distribution behaves similarly to random graphs.

Definition 2 (Spectral Expander Graphs.)**.** A sequence of d-regular graphs is called a *Family of Spectral Expander Graphs* when for each graph λ is significantly smaller than d.

Connection. Define $d - \lambda_2$ to be the *Spectral Gap* of the graph. Cheeger's Inequality establishes a deep connection between combinatorial expansion and spectral properties of graphs.

Lemma 2 (Cheeger's Inequality)**.** *For every d-regular graph,*

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Hence a large spectral gap implies strong combinatorial expansion. Equivalently, graphs with smaller λ_2 are better expanders.