Mathematics for Computer Science #1

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Please feel free to contact in case of any mistakes, comments, or feedback.

1 Set

In this section, we will define a set and some concerning notations.

Definition 1.1 (Set)**.** *A set is an* unordered *collection of* distinct *elements. For example:* $\{1, 2, 3\}$*, set of even numbers, etc.*

A set can be defined implicitly or explicitly.

Explicit Definition. A set can be defined explicitly by listing its each element. For example: $\{1, 2, 3\}$.

Implicit Definition. A set can be defined implicitly by describing its constructive property. For example: $\{x \mid x > 1\}$.

The following symbols are used to denote special sets.

1. $\mathbb{N}$: Natural Numbers $(1, 2, 3, \dots)$
2. $\mathbb{Z}$: Integers $(\dots, -2, -1, 0, 1, 2, \dots)$
3. $\mathbb{Q}$: Rational Numbers
4. $\mathbb{R}$: Real Numbers
5. ϕ: $\{\}$

Cardinality. The number of elements in the set is called its cardinality and denoted by $|A|$. Based on cardinality, we can classify sets into *finite* and *infinte*. Now, we define subset and power set of a set.

Definition 1.2 (Subset)**.** *A set A is called a subset of B if* $\forall x(x \in A \implies x \in B)$*.*

Definition 1.3 (Power set). *Let A be a set. Power set of A is defined to be the set $P = \{B \mid B \text{ is a subset of } A\}$.*

Remark 1.4. *Let A be a set such that $|A| = n$ and P is power set of A. Then, the following is true:*

1. $\phi \in P$.
2. $|P| = 2^n$.

Definition 1.5 (Equality.). *Let A and B be two sets. Then, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

2 Set Operations

We have the following fundamental operations on sets. Let U be the *universe* set $A, B \subseteq U$ be two sets over U and define

1. *Union* $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$
2. *Intersection* $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$
3. Difference $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$
4. *Complement* $A^C := \{x \mid x \notin A\}$
5. *Symmetric Difference* $A \Delta B := (A \setminus B) \cup (B \setminus A)$

3 De Morgan's Laws

Theorem 3.1 (De Morgan's Laws). *Let A and B be two sets. Then the following is true:*

1. $(A \cup B)^C = A^C \cap B^C$
2. $(A \cap B)^C = A^C \cup B^C$

Proof. (i) $(A \cup B)^C = A^C \cap B^C$.

1. $(A \cup B)^C \subseteq A^C \cap B^C$.
   Let $x \in (A \cup B)^C$. Then,

$$
\begin{aligned}
&x \notin A \cup B \\
&x \notin A \text{ and } x \notin B \\
&x \in A^C \cap B^C
\end{aligned}
$$

2. $A^C \cap B^C \subseteq (A \cup B)^C$.
   Let $x \in A^C \cap B^C$. Then,

$$\begin{aligned}
&x \in A^C \text{ and } x \in B^C \\
&x \notin A \text{ and } x \notin B \\
&x \notin A \cup B \\
&x \in (A \cup B)^C
\end{aligned}$$

(ii) $(A \cap B)^C = A^C \cup B^C$.

1. $(A \cap B)^C \subseteq A^C \cup B^C$.
   Let $x \in (A \cap B)^C$. Then,

$$x \notin A \cap B$$

   (a) Case $(x \in A)$. Since $x \notin A \cap B$, thus,

$$x \in B^C.$$

   (b) Case $(x \in A^C)$. Then,

$$x \in A^C \cup B^C.$$

2. $A^C \cup B^C \subseteq (A \cap B)^C$.
   Let $x \in A^C \cup B^C$.

   (a) Case $((x \in A^C))$. Then,

$$\begin{aligned}
&x \notin A \cap B \\
&x \in (A \cap B)^C
\end{aligned}$$

   (b) Case $(x \in B^C)$. Then,

$$\begin{aligned}
&x \notin B \cap A \\
&x \in (A \cap B)^C
\end{aligned}$$

□

4 Distribution Laws

Theorem 4.1 (Distribution Laws)**.** *Let A, B, C be three sets. Then, the following is true:*

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
   Let $x \in A \cup (B \cap C)$. Then,

   (a) Case ($x \in A$). Then,
   $$x \in A \cup B \text{ and } x \in A \cup C$$
   $$x \in (A \cup B) \cap (A \cup C)$$

   (b) Case ($x \in B \cap C$). Then,
   $$x \in B \cup A \text{ and } x \in C \cup A$$
   $$x \in (A \cup B) \cap (A \cup C)$$

2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
   Let $x \in (A \cup B) \cap (A \cup C)$. Then,

   (a) Case ($x \in A$). Then,
   $$x \in A \cup (B \cap C)$$

   (b) Case $x \notin A$. Then,
   $$x \in B \text{ and } x \in C$$
   $$x \in (B \cap C)$$
   $$x \in A \cup (B \cap C)$$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ Let $x \in A \cap (B \cup C)$. Then
   $$x \in A$$
   .

   (a) Case ($x \in B$). Then,
   $$x \in A \cap B$$
   $$x \in (A \cap B) \cup (A \cap C)$$

   (b) Case ($x \in C$). Then,
   $$x \in A \cap C$$
   $$x \in (A \cap B) \cup (A \cap C)$$

2. $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
   Let $x \in (A \cap B) \cup (A \cap C)$. Then,

(a) Case ($x \in A \cap B$). Then,

$$\begin{aligned} x \in A &\text{ and } x \in B \\ x &\in B \cup C \\ x &\in A \cap (B \cup C) \end{aligned}$$

(b) Case ($x \in A \cap C$). Then,

$$\begin{aligned} x \in A &\text{ and } x \in C \\ x &\in C \cup B \\ x &\in A \cap (B \cup C) \end{aligned}$$

$\square$

5 Partition and Cartesian Product

Definition 5.1 (Partition)**.** *Let A be a set. A sequence of sets $A_1, A_2, \ldots, A_k$ for some $k \in \mathbb{N}$ is called a partition of A if*

1. *For all i, j, $A_i \cap A_j = \phi$*
2. $\bigcup_{i \in [k]} A_i = A$

Theorem 5.2 (Sum Rule)**.** *Let $A_1, \ldots, A_k$ be a partition of the set A. Then,*

$$|A| = |A_1| + |A_2| + \cdots + |A_k|$$

Definition 5.3 (Cartesian Product)**.** *Let A, B be two sets. The* Cartesian *product of A and B is defined as,*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

where $(a_1, b_1) = (a_2, b_2)$ iff $a_1 = a_2$ and $b_1 = b_2$.

Remark 5.4. *The Euclidean plane is the cartesian product $\mathbb{R} \times \mathbb{R}$.*

Theorem 5.5 (Product Rule)**.** *Let A, B be two sets. Then,*

$$|A \times B| = |A| \cdot |B|$$

Proof. Consider the partition of $P = A \times B$:

$$P_a = \{(a, b) \mid b \in B\}$$

for every $a \in A$. Note that

$$\bigcup_{a \in A} P_a = P$$

Using Theorem 5.2,

$$|P| = \sum_{a \in A} |P_a| = |A| \cdot |B|$$

$\square$